Phys141 – Mon 10/23

Chapter 10 - 12
Moment of inertia
Angular momentum
Static equilibrium

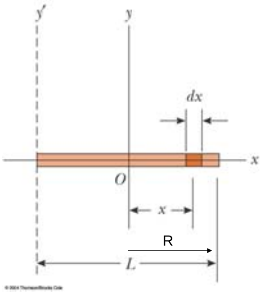
Moment of Inertia of a Uniform Rigid Rod

- The shaded area has a mass
  $dm = M/L\,dx$
- Then the moment of inertia is

$$I = \int r^2 dm = \int_{-L/2}^{L/2} x^2 \frac{M}{L} dx$$

$$I = \frac{1}{3}\frac{M}{L} x^3 \Big|_{-L/2}^{L/2} = \frac{1}{12} M L^2$$

Note: Careful about the choice of origin. That should be the point of rotation



Parallel-Axis Theorem

- Previous examples, axis of rotation ~ axis of symmetry of the object and therefore the axis of rotation went through the center of mass.
- Arbitrary axis -> integral difficult -> the parallel-axis theorem helps:

$$I = I_{CM} + MD^2$$

  - I is about any axis **parallel** to the axis through the center of mass of the object
  - I_{CM} is about the axis through the center of mass
  - D is the distance from the center of mass axis to the arbitrary axis

Moment of Inertia for a Rod Rotating Around One End

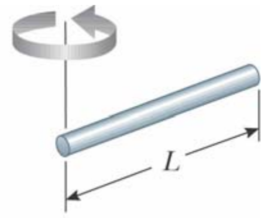
- The moment of inertia of the rod about its center is

$$I_{CM} = \frac{1}{12}ML^2$$

- Distance between center and end of rod is ½ *L*

$$I = I_{\text{CM}} + M\left(\frac{L}{2}\right)^2$$

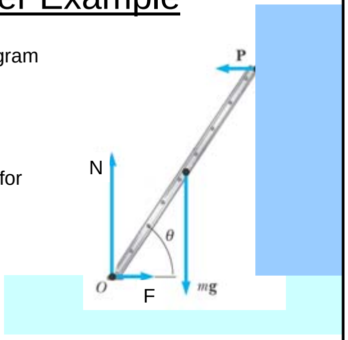
$$I = \frac{1}{12}ML^2 + M\left(\frac{L}{2}\right)^2 = \frac{1}{3}ML^2$$



Ladder Example

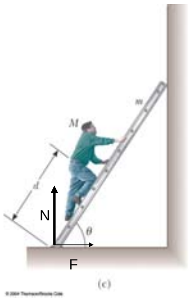
- Draw a free-body diagram for the ladder (see on right)
- Choose *O* as axis of rotation (fewer forces generate torque than for point 1 -> easier calculation)

-> Two conditions of equilibrium (solve on blackboard)



Ladder Example, Extended

- Add a person of mass *M* at a distance *d* from the base of the ladder
- The higher the person climbs, the larger F is relative to N
- Eventually, the ladder may slip



Torque and Angular Acceleration

- Rotating mass *m* (along circle of radius *r)* tangential force $\mathbf{F}_t$

-> tangential acceleration:

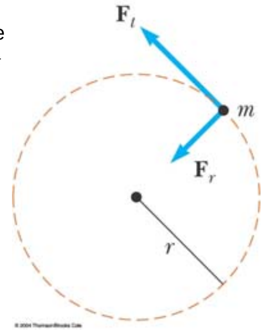
$F_t = ma_t$

Rotational motion description:

$$\tau = F_t r = ma_t r = mr^2 \frac{a_t}{r} = I\alpha$$

$$\tau = I\alpha$$

In general: $\Sigma\tau = I\alpha$

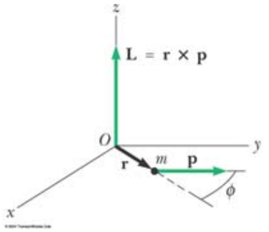


Angular Momentum

Angular momentum **L** relative to the origin *O:*

Cross product of the particle position **r** and linear momentum **p**

L = **r** x **p**



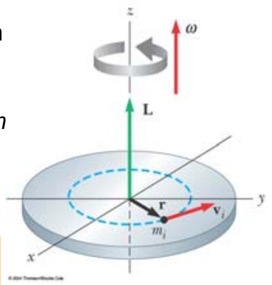
Angular Momentum of a Rotating Rigid Object

Each particle of the object rotates in the *xy* plane about the *z* axis with an angular speed of ω

Angular momentum of an individual particle: $L_i = n$ $r_i^2\ \omega$

Sum over all particles:

$$L_z = \sum_i L_i = \sum_i m_i r_i^2 \omega = I\omega$$



Conservation of Angular Momentum

Position of masses of an isolated system changes (w/o external forces):

-> moment of inertia changes

-> Conservation of angular momentum requires a compensating change in the angular velocity

$$I_i\,\omega_i = I_f\,\omega_f$$

- **Demo: chair**

Torque and Angular Momentum

- The torque is related to the angular momentum

$$\boxed{\sum \tau = \frac{d\mathbf{L}}{dt}} = \mathbf{r}\times\frac{d\mathbf{p}}{dt} + \frac{d\mathbf{r}}{dt}\times\mathbf{p}$$

- Rotational analog of Newton’s Second Law
  - $\Sigma\tau$ and **L** must be measured **about the same origin**
  - This is valid for **any origin fixed in an inertial frame**

No external torque (isolated system):
angular momentum conserved

$$I_i\,\omega_i = I_f\,\omega_f$$

Conservation Law Summary

- For an isolated system -

(1) Conservation of Energy:

- $E_i = E_f$

(2) Conservation of Linear Momentum:

- $\mathbf{p}_i = \mathbf{p}_f$

(3) Conservation of Angular Momentum:

- $\mathbf{L}_i = \mathbf{L}_f$

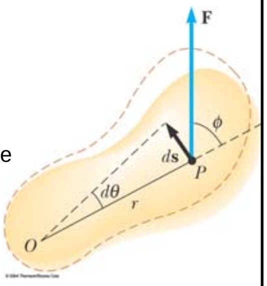
Work in Rotational Motion

Work done by **F** on the object as it rotates through an infinitesimal distance $ds = r\,d\theta$

$dW = \mathbf{F} \cdot d\,\mathbf{s} = (F \sin \phi)\, r\, d\theta$

$dW = \tau\, d\theta$

Note: Both work and torque have units of Nm – but remember: torque is not an energy!!



Power in Rotational Motion

- The rate at which work is being done in a time interval *dt* is

$$\text{Power} = \frac{dW}{dt} = \tau \frac{d\theta}{dt} = \tau\omega$$