Phys141 – Wed 10/18 – Lecture 20

Today: Chapter 10

Administrative:
Reading for Fri: chapter 11.1-4

Moment of Inertia

Rotational energy

$$K_R = \sum_i K_i = \sum_i \frac{1}{2} m_i r_i^2 \omega^2$$

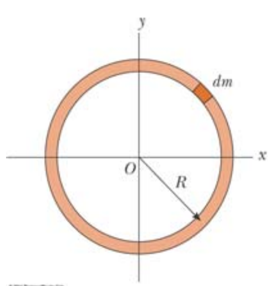
$$K_R = \frac{1}{2}\left(\sum_i m_i r_i^2\right)\omega^2 = \frac{1}{2} I \omega^2$$

- Definition of moment of inertia: $I = \sum_i r_i^2 m_i$
- Dimensions: ML^2
- SI units: $kg{\cdot}m^2$
- Calculation: $I = \lim_{\Delta m_i \to 0} \sum_i r_i^2 \Delta m_i = \int r^2 dm$

Moment of Inertia of a Uniform Thin Hoop

- Since this is a thin hoop, all mass elements are the same distance from the center

$$I = \int r^2 dm = R^2 \int dm$$
$$I = MR^2$$



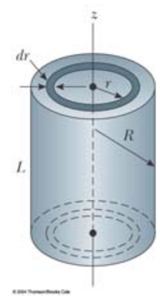
Moment of Inertia of a Uniform Solid Cylinder

- Divide the cylinder into concentric shells with radius *r*, thickness *dr* and length *L*
- Then for *I*

$$I = \int r^2 dm = \int r^2 \left(2\pi\rho L r \, dr\right)$$

$$I_z = \frac{1}{2} M R^2$$

DEMO



Total Energy – ball rolling down hill

- Despite surface friction, no loss of mechanical energy occurs because the contact point is at rest relative to the surface at any instant

$K_f + U_f = K_i + U_i$

$U_f = K_i = 0$

$$K_f = \frac{1}{2} I_{CM} \omega^2 + \frac{1}{2} M v_{CM}^2$$

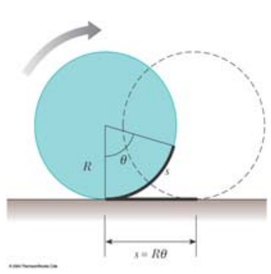
Connecting rotation and COM motion

- The velocity of the center of mass:

$$v_{CM} = \frac{ds}{dt} = R\frac{d\theta}{dt} = R\omega$$

The acceleration of the center of mass is

$$a_{CM} = \frac{dv_{CM}}{dt} = R\frac{d\omega}{dt} = R\alpha$$



Total Energy – ball rolling down hill

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$K_f + U_f = K_i + U_i$

$U_f = K_i = 0$

$$K_f = \frac{1}{2} I_{CM}\omega^2 + \frac{1}{2} M v_{CM}^2 = \frac{1}{2} M v_{CM}^2 \left(\frac{I_{CM}}{MR^2} + 1 \right)$$

$U_i = Mgh$

$$Mgh = \frac{1}{2} M v_{CM}^2 \left(\frac{I_{CM}}{MR^2} + 1 \right) \rightarrow v_{CM}^2 = \frac{2gh}{\left(\frac{I_{CM}}{MR^2} + 1 \right)}$$

Moment of inertia in terms of densities

Calculate inertia by integrating over length, area, or volume instead of mass:

$$I = \int r^2 dm$$

Linear Mass Density
mass per unit length of a rod of uniform cross-sectional area A:

$$\lambda = \frac{m}{L} = \rho A \qquad I = \int \lambda r^2 dL$$

Area mass density:
Mass per unit area of a sheet of thickness L

$$\sigma = \frac{m}{A} = \rho L \qquad I = \int \sigma r^2 dA$$

Volumetric Mass Density
mass per unit volume:

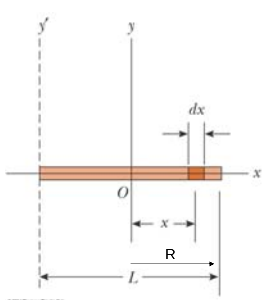
$$\rho = \frac{m}{V} \qquad I = \int \rho r^2 dV$$

Moment of Inertia of a Uniform Rigid Rod

- The shaded area has a mass
  $dm = M/L\,dx$
- Then the moment of inertia is

$$I = \int r^2 dm = \int_{-L/2}^{L/2} x^2 \frac{M}{L} dx$$

$$I = \frac{1}{3}\frac{M}{L} x^3 \Big|_{-L/2}^{L/2} = \frac{1}{12} ML^2$$



Note: Careful about the choice of origin. That should be the point of rotation

Parallel-Axis Theorem

- Previous examples, axis of rotation ~ axis of symmetry of the object and therefore the axis of rotation went through the center of mass.
- Arbitrary axis -> integral difficult -> the parallel-axis theorem helps:

$$I = I_{CM} + MD^2$$

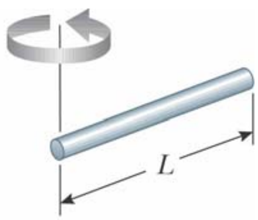
- – I is about any axis **parallel** to the axis through the center of mass of the object
- – I_{CM} is about the axis through the center of mass
- – D is the distance from the center of mass axis to the arbitrary axis

Moment of Inertia for a Rod Rotating Around One End

- The moment of inertia of the rod about its center is

$$I_{CM} = \frac{1}{12}ML^2$$

- Distance between center and end of rod is ½ L



$$I = I_{CM} + M\left(\frac{L}{2}\right)^2$$

$$I = \frac{1}{12}ML^2 + M\left(\frac{L}{2}\right)^2 = \frac{1}{3}ML^2$$

Torque: tendency of a force to rotate an object about some axis

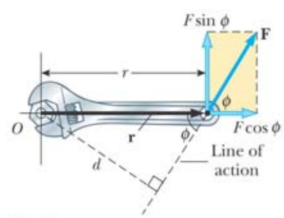
Magnitude: $\tau = r F \sin \phi$

Two ways to understand torque equation:

(1) $\tau = F d$

d: *perpendicular* distance from the axis of rotation to a line drawn along the direction of the force

$d = r \sin \Phi$



(2) $\tau = F_t r$

F_t: tangential part of force

$F_t = F \sin \Phi$

Net Torque

F_1 would cause counter-clockwise rotation about *O*

F_2 would cause clockwise rotation about *O*

Total (net) torque = sum of torques

$\Sigma\tau = \tau_1 + \tau_2 = F_1 d_1 - F_2 d_2$