PHYSICS OF PLASMAS VOLUME 8, NUMBER 12 DECEMBER 2001

Convection in an asymmetrically sourced Z pinch

A. M. Rey and A. B. Hassam
Institute for Research in Electronics and Applied Physics, University of Maryland, College Park, Maryland 20742

(Received 27 July 2001; accepted 30 August 2001)

The convection of a magnetically confined plasma resulting from heat and particle sources is studied. It is assumed that the convection is low-level in that the system stays stable to ideal interchange instabilities. A Z-pinch plasma with asymmetric particle and heat sources is considered. It is found that there is no convection if there are no particle sources, independent of the distribution of the heat sources. Particle sources result in convection which in turn influences heat transport. The central temperature, however, may go up or down in response to this convection, depending on the distribution of the source function.
[DOI: 10.1063/1.1413228]

I. INTRODUCTION

In a gravitational field, fluids and plasmas settle into convection patterns when heated. The nature of the convection is dramatically different depending on whether the isotherms are linearly stable to the attendant Rayleigh–Taylor mode. If the heat sources create profiles that are linearly unstable, the system is turbulent and the resulting convection transports heat rapidly away from the heat sources. If the profiles are linearly stable, however, the nature of the convection is different: If the heating is uniform in the directions transverse to the gravitational field, there is no convection and heat flows by laminar, classical thermal conduction; if the heating is nonuniform, there are convection cells, in general, and these cells are expected to transport heat although the transport rate is slower than the (unstably) turbulent rate above.

The latter type of convection, convection in a stable system from asymmetric heating, has been of revived interest in magnetized plasmas.[1–7] In a magnetized plasma, the counterpart of the Rayleigh–Taylor mode is the interchange instability, wherein curvature in the magnetic field mimics the gravitational field in fluids.[8] A recent experiment in magnetic fusion confinement (the Levitated Dipole Experiment, LDX[9]) has revived interest in stable convection. In this experiment, plasma is created in a dipolar magnetic field. The system is made large enough so that the pressure gradient of the plasma is relatively gentle when compared with the rapid fall off of the magnetic-field strength away from the dipolar coil. Consequently, the interchange mode is stable, in much the same way as the inner radial layers of the sun are stable in spite of central heating. The LDX, however, will likely be heated nonaxisymmetrically. This suggests that there would be stable convection in the system, as was indeed seen in related earlier experiments.[2–4] If so, it is desirable to understand the origin and nature of the convection. In particular, how much heat and particles will the cells transport and how can the heaters be adjusted to minimize any deleterious effects on the transport? (Convection may also result from asymmetric sources of particles.) In this paper, we address some of these questions. Kesner and Garnier[1] have also recently studied this problem. We discuss their results of this study in the context of our findings in Sec. V herein.

For simplicity, we first consider a simpler magnetic geometry—a linear Z pinch with a central current carrying rod. This is equivalent to the dipolar system examined close to the current coil. We assume that the Z pinch is in the interchange-stable regime and study the convection resulting from asymmetric sourcing. The study yields some unexpected results: First, that heat sources do not result in convection, only particle sources do; second, that in certain cases the heat flux from the convection can be negative, i.e., towards the high temperature zones.

In the next section, we set up the equations that apply to this quasistatic system. In Sec. III, we discuss some general considerations and check the simple limits of the system. In Sec. IV, we study the full system. We conclude in Sec. V.

II. EQUATIONS

The equations governing a Z pinch in quasistatic steady state are[10]

$$\nabla\cdot n\boldsymbol{u}=S, \tag{1}$$

$$\nabla(p+B^2/2)-\boldsymbol{B}\cdot\nabla\boldsymbol{B}=0, \tag{2}$$

$$\boldsymbol{E}+\boldsymbol{u}\times\boldsymbol{B}-\eta\nabla\times\boldsymbol{B}=0, \tag{3}$$

$$\nabla\times\boldsymbol{E}=0, \tag{4}$$

$$\boldsymbol{u}\cdot\nabla p+\gamma p\nabla\cdot\boldsymbol{u}=(\gamma-1)\nabla\cdot(\kappa\nabla T)+H+TS, \tag{5}$$

$$p\equiv 2nT. \tag{6}$$

Standard notation is used. S is the particle source, H is the heat source, M is the ion mass, and κ is the thermal conductivity. We assume ions and electrons have the same temperature T, and therefore, the same pressure due to the quasineutrality of the plasma. The total pressure is denoted by p. The center of mass velocity is $\boldsymbol{u}$, and the current density $\boldsymbol{J}=\nabla\times\boldsymbol{B}$. These equations are valid in the quasistatic approximation, $|\boldsymbol{u}|\ll c_s$, where $c_s\equiv(T/M)^{1/2}$ is the sound speed. For simplicity, we have ignored the gyroviscosity in Eq. (2); this shortcoming will be discussed later. We assume that there is no momentum source and, thus, a source term does not appear in the momentum balance. Since we have also neglected gyroviscous terms, the absence of momentum sources means that there will be no average axial flow in equilibrium. (Collisional viscosity brings the flow to rest, assuming coupling to the walls. As mentioned, we will discuss gyroviscous effects later.) Thus, we assume for simplicity that there is to average axial flow at any radius. We also assume that particles are sourced into the system at a temperature equal to the ambient temperature of the plasma. Thus, if the heat source H is zero, the local temperature does not change but the density and pressure change if S is nonzero. This is the reason why the term TS appears in Eq. (5). We note that if the particles are sourced in at low temperature, the TS term in Eq. (5) may be neglected. There is also a term proportional to SMu^2 in this equation that we have neglected since it is small in the quasistatic, subsonic limit we consider here. Consistent with the neglect of gyroviscosity, we also have neglected drift heat flux terms.

For simplicity, we will assume that $H=H(r,z)$ and $S=S(r,z)$. This is justified on account of the rapid thermal communication along the field lines. Here, (r,θ,z) are the standard cylindrical coordinates. Thus, we will look for an equilibrium where θ is ignorable and there is no azimutal flow, $u_\theta=0$.

A set of equations applied to the Z pinch is arrived at as follows. We let $\boldsymbol{B}=\hat{\theta}B$, in which case the z-component of Eq. (2) yields

$$p+B^2/2=f(r), \tag{7}$$

whereupon the radial component yields

$$df/dr=-B^2/r. \tag{8}$$

Equations (7) and (8) imply that $p=p(r)$ and $B=B(r)$, and so Eq. (8) is rewritten as

$$\frac{d}{dr}\left(p+\frac{B^2}{2}\right)=-\frac{B^2}{r}. \tag{9}$$

Now consider Eq. (3). Given the forms for p and $\boldsymbol{B}$ as found above and the azimuthal symmetry assumed, we find that $E_\theta\equiv 0$. Then, the only nontrivial component of Eq. (4) is the θ component

$$\partial E_r/\partial z=\partial E_z/\partial r. \tag{10}$$

We split this into its z-averaged part and a part fluctuating in z. The averaged piece is $\partial \overline{E_z}/\partial r=0$, where $\overline{g}=\oint dz g/\oint dz$. Integration yields $\overline{E_z}=\text{const}$, which can be set to zero because we will not have an externally applied electric field in this problem. The resulting equation, upon inserting $\overline{E_z}=0$, is

$$\overline{u_r B}=\frac{\eta}{r}\frac{d}{dr}(rB). \tag{11}$$

The fluctuating piece of Eq. (10) becomes, using Eq. (3)

$$\frac{\partial}{\partial r}(B\widetilde{u_r})+B\frac{\partial}{\partial z}\widetilde{u_z}=0, \tag{12}$$

where $\tilde{g}\equiv g-\overline{g}$.

We now consider Eqs. (1) and (5). The z-average of (1) yields

$$\frac{1}{r}\frac{d}{dr}[r(\overline{nu_r}+\overline{\tilde{n}\ \tilde{u}_r}]=\overline{S}, \tag{13}$$

and the z-average of Eq. (5) yields

$$\overline{u_r}\frac{dp}{dr}+\frac{\gamma p}{r}\frac{d}{dr}(r\overline{u_r})=\frac{\gamma-1}{r}\frac{d}{dr}\left[\overline{r\kappa\frac{dT}{dr}}\right]+\overline{H}+\overline{TS}+\overline{\tilde{T}\tilde{S}}. \tag{14}$$

The fluctuating part of Eq. (1) is

$$\frac{1}{r}\frac{\partial}{\partial r}[r(\overline{n}\tilde{u}_r+\overline{u_r}\tilde{n}+\widetilde{\tilde{n}\tilde{u}_r})]+\frac{\partial}{\partial z}[\overline{n}\tilde{u}_z+\widetilde{\tilde{n}\tilde{u}_z}]=\tilde{S}, \tag{15}$$

where we have used that axial average flux is zero, $\overline{u_z}=0$.

The fluctuating part of Eq. (5) is

$$p\left[\frac{1}{p}\frac{dp}{dr}+\frac{\gamma B}{r}\frac{d}{dr}\left(\frac{r}{B}\right)\right]\tilde{u}_r=(\gamma-1)\boldsymbol{\nabla}\cdot(\kappa\tilde{\boldsymbol{\nabla}}T)+\tilde{H}+\overline{T}\tilde{S}+\overline{S}\tilde{T}+\widetilde{\tilde{S}\tilde{T}}, \tag{16}$$

where in Eq. (16) we have used Eq. (12).

Equations (6), (9), and (11)–(16) constitute nine equations for the nine variables p, B, $\overline{n}$, $\tilde{n}$, $\overline{T}$, $\tilde{T}$, $\tilde{u}_r$, $\tilde{u}_z$, and $\overline{u_r}$. [Equation (6) actually represents two equations.] The system describes quasistatic convection resulting from asymmetric sources. The flow can be decomposed as an average radial flow, $\overline{u_r}$, and flows fluctuating in z, $\tilde{u}_r$, and $\tilde{u}_z$. We will refer to the latter flows as "convective cells" or "convective flows."

It is noteworthy that while the average radial flows depend on resistivity [see Eq. (11)], the convective flows do not. The latter can be seen from Eq. (12) which, in fact, expresses the fact that $\tilde{\boldsymbol{E}}=-\tilde{\boldsymbol{u}}\times\boldsymbol{B}$, that is to say the plasma in the convective cells is frozen-in to the flux tubes. [The reason for this, as can be seen from Eq. (3), is that $\tilde{\boldsymbol{J}}=0$, since $B=B(r)$.] This finding has a general important consequence, namely, the convective cells cannot transport particles. As a flux tube moves, it carries enclosed particles with it and the particle density may compress or decompress as the tube "pumps" up or down; however, no particles will diffuse out of the flux tube and, in a closed convective loop where all the tubes interchange positions, the particles will return to where they began with their numbers intact. Of course, particles may be born into a tube or be destroyed, from the sources S. Note also that while particles may not diffuse out of tube, heat can. Thus, in a closed convective loop, heat can be lost during one complete circuit. We thus expect convective cells to contribute to heat transport. These aspects will be discussed later.

Equation (16) in the above set is also particularly noteworthy: Note that if the expression in the brackets on the left-hand side (lhs) goes to zero, $\tilde{u}_r$ diverges. This is because the bracketed expression is just the stability criterion for the ideal interchange instability[8] (stability if the expression is positive). Thus, as we approach the marginal point for interchange stability, the convection gets larger and larger. Clearly, at some point the quasistatic assumption fails and instability sets in. We discuss this interesting by-product of our calculation in more detail later.

III. SIMPLE LIMITS

We will now study the behavior of this system in certain limits before analyzing the general case.

A. Axial symmetry

The simplest case is the axially symmetric limit, i.e., when $\tilde{H}$ and $\tilde{S}$ are zero. Then $\tilde{n}$, $\tilde{\boldsymbol{u}}$, and $\tilde{T}$ can be set to zero, satisfying identically Eqs. (12), (15), and (16). The remaining equations are

$$\frac{dp}{dr}+B\frac{dB}{dr}=-\frac{B^2}{r}, \tag{17}$$

$$\overline{u}_r=\eta\frac{1}{r}\frac{d}{dr}(rB), \tag{18}$$

$$\frac{1}{r}\frac{d}{dr}(r\overline{nu_r})=\overline{S}, \tag{19}$$

$$\overline{u}_r\frac{dp}{dr}+\frac{\gamma p}{r}\frac{d}{dr}(r\overline{u}_r)=\frac{\gamma-1}{r}\frac{d}{dr}\left[r\kappa\frac{d\overline{T}}{dr}\right]+\overline{H}+\overline{TS}, \tag{20}$$

four equations that govern $\overline{n}$, $\overline{T}$, B, and $\overline{u}_r$ with $p=2nT$.

Consider the special case $\overline{S}=0$. Then, Eq. (19) yields $\overline{rnu_r}=\text{const}$. Suppose we assume that there is no mass loss at the wall, i.e., $\overline{u}_r(r=a)=0$. Then, the constant is zero and $\overline{u}_r=0$ everywhere. In that case, Eq. (18) yields $B=\text{const}/r$, i.e., the field relaxes to the vacuum field due to current carrying core. Basically, Eq. (18) represents a balance between frozen-in convection of B from any $\overline{u}_r$ and resistive diffusion of B (current relaxation). If there are no particle sources, frozen-in convection is zero and the magnetic field resistively relaxes to the vacuum field. But this means that pressure gradients cannot be maintained, i.e., $p\rightarrow\text{const}$ as seen from Eq. (17). To summarize then, axial symmetry and $\overline{S}=0\Rightarrow\overline{u}_r=0$, $B\rightarrow C/r$, $p\rightarrow\text{const}$ and the temperature is given by

$$\frac{\gamma-1}{r}\frac{d}{dr}\left[r\kappa\frac{d\overline{T}}{dr}\right]=-\overline{H}, \tag{21}$$

with $p=2nT$.

If $\overline{S}\neq 0$, there is a radial flow, the B field deviates from vacuum, driven by this flow, and the pressure profile adjusts to stay in force balance, according to Eq. (17). Thus, particle sources determine pressure profiles, which would otherwise be flat.

Note that we have used the boundary condition that $u_r(r=a)=0$, i.e., we assume that whatever particles are lost to the walls are assumed to eventually reenter the plasma. This boundary condition is not as restrictive as it appears, though. This is because the form for S assumed in this paper is quite general; in particular, $S(r,z)$ could be assumed to have a sink localized at the walls which could model particle extraction at the walls. The only restriction on S is that the volume integral vanish in accordance with the steady state assumption for the whole calculation.

B. Axially dependent heating, no particle sources

Suppose $S=0$ but H is arbitrary, not necessarily axially symmetric. From the above system, we can show that a solution exists such that $\boldsymbol{u}=0$. That is to say, nonaxisymmetric heating is not sufficient to drive convection cells if there are no particle sources. To see this, we note that Eqs. (13), (12), and (15) are identically satisfied. Next, Eq. (11) is satisfied if $B\propto 1/r$, whereupon Eq. (9) yields $p=\text{const}$. The remaining equations are Eqs. (14) and (16) which become

$$\frac{\gamma-1}{r}\frac{d}{dr}\left[r\kappa\frac{d\overline{T}}{dr}\right]=-\overline{H}, \tag{22}$$

$$(\gamma-1)\boldsymbol{\nabla}\cdot[\kappa\tilde{\boldsymbol{\nabla}}T]=-\tilde{H}. \tag{23}$$

Equations (22) and (23) constitute a complete set for $\overline{n}$, $\tilde{n}$, $\overline{T}$, and $\tilde{T}$ together with the definition $p=2nT$. Thus, a solution with the magnetic field, the vacuum field, the pressure a constant, n and T dependent on z consistent with H, and $\boldsymbol{u}=0$ can be found if $S=0$.

IV. GENERAL CASE

A. Weak sources

The general case with $S\neq 0$ and with H arbitrary is difficult to solve analytically. We first try the simple case in which S and $\tilde{H}$ are small perturbations but $\overline{H}$ is not small.

Using a perturbative approach, the lowest order solution is the one given in Sec. III A, i.e., $\boldsymbol{u}_0=0$, $B_0\propto 1/r$, $p_0=\text{const}$, and T_0 given by Eq. (21). Going to first order, we get from Eq. (13) an expression for $\overline{u}_{r1}$, which can be used in Eq. (11) to get B_1 and in Eq. (14) to get $\overline{T}_1$. Using B_1 in Eq. (9) we get p_1, that together with $\overline{T}_1$ determines $\overline{n}_1$. Regarding the fluctuating quantities, we see that Eq. (15) determines $\tilde{u}_{r1}$ whereupon Eq. (16) determines $\tilde{T}_1$, expressions that can be used in Eqs. (12) and (6), respectively, to find $\tilde{u}_{z1}$ and $\tilde{n}_1$.

All told, the first-order solutions are

$$\overline{u}_{r1}=\frac{\int_0^r\overline{S}(r')r'dr'}{rn_0}, \tag{24}$$

$$B_1=\frac{1}{r}\int_0^r\frac{B_0(r')\overline{u}_{r1}(r')r'}{\eta}dr', \tag{25}$$

$$p_1=\int_r^a\frac{B_0^2(r')}{\eta}\overline{u}_{r1}(r')dr', \tag{26}$$

$$\overline{T}_1 = -\frac{\gamma}{\gamma-1} p_0 \int_r^a \frac{\overline{u_{r1}(r')}}{\kappa} dr' + \frac{1}{\gamma-1} \int_r^a \int_0^{r'} \frac{r'' T_0(r'') \overline{S}(r'')}{\kappa r'} dr' dr'', \tag{27}$$

$$\overline{n}_1 = \frac{p_1 - 2 n_0 \overline{T}_1}{2 T_0}, \tag{28}$$

$$\nabla \cdot (\kappa \tilde{\nabla} T_1) = \frac{2\gamma}{\gamma-1} \frac{p_0}{r} \frac{\tilde{S}}{D(r)} - \frac{\tilde{H} + T_0 \tilde{S}}{\gamma-1}, \tag{29}$$

$$\tilde{u}_{r1} = \frac{\tilde{S}}{D(r)}, \tag{30}$$

$$\tilde{u}_{z1} = -\frac{1}{B_0} \frac{\partial}{\partial r} \left(B_0 \left(\int_0^z \tilde{u}_{r1} dz + \overline{\tilde{u}_{r1} z} \right) \right), \tag{31}$$

$$\tilde{n}_1 = -\frac{n_0 \widetilde{T_1}}{T_0}, \tag{32}$$

where

$$D(r) \equiv \frac{dn_0}{dr} + \frac{2 n_0}{r}. \tag{33}$$

We assume that the boundary conditions are determined by the zero-order quantities, so all higher order variables are set to zero at the boundaries.

From Eqs. (24)–(32), we discern the following:

(a) All the averaged first order quantities, $(\overline{u}_{r1}, \overline{n}_1, \overline{T}_1, B_1, p_1)$, are driven by the averaged sources $\overline{S}$ and $\overline{H}$ but are independent of $\tilde{S}$ and $\tilde{H}$.

(b) All the fluctuating first order quantities, $(\tilde{u}_{r1}, \tilde{u}_{z1}, \tilde{n}_1, \tilde{T}_1)$, are driven by $\tilde{S}$ and $\tilde{H}$, as is evident from Eqs. (29)–(32). In particular, convective flows are driven only by the asymmetric sources $\tilde{S}$ which in turn drive temperature and density fluctuations.

(c) The pressure profile is determined by $\overline{S}$. From Eq. (26), we see that if the averaged radial flow is outward, $dp_1/dr<0$, and vice versa. If $\int^r r dr S$ is positive, there is an outward flow that convects the magnetic flux with it, to be mitigated by flux diffusing back in. Thus, $dB/dr>0$ and $dp_1/dr<0$ for force balance.

(d) The average temperature change, $\overline{T}_1$, is not influenced by the convective cells: it depends only on $\overline{S}$, not $\tilde{S}$.

Our central concern in this paper is to understand if and how convection transports particles and heat. Let us thus examine particle and heat fluxes. Integrating the averaged equation of continuity, Eq. (13), we get an expression for the average particle flux, $\overline{\Gamma}_p \equiv \overline{n u_r}$, viz.

$$\overline{\Gamma}_p = \frac{\int_0^r \overline{S}(r') r' dr'}{r}. \tag{34}$$

Using Eq. (24) into Eq. (34) we get

$$\overline{\Gamma}_{p1} = n_0 \overline{u}_{r1},$$
$$\overline{\Gamma}_{pn} = 0, \tag{35}$$

where $\overline{\Gamma}_{pn}$ is the particle flux to nth order. Thus, beyond the first order, the particle flux is zero to all orders. Furthermore, all the flux comes from the average flows, not from the convective cells, in accordance with our discussion earlier on the frozen-in nature of the flow. Since Eq. (34) does not assume a small S, the conclusion that convective cells do not transport particles is general.

We now consider if heat flux could be carried by the convective cells. We do this by examining changes in the average central temperature, $\overline{T}$, resulting from convection. As shown above, convective cells do not affect the average central temperature to first order. Thus, we need to go to higher order. To second order, Eq. (14) takes the form

$$\overline{\kappa \frac{dT_2}{dr}} = \frac{1}{\gamma-1} \left(p_1 \overline{u}_{r1} + \frac{\gamma-1}{r} \int_0^r p_1 \frac{d(r' \overline{u}_{r1})}{dr'} dr' \right) + \frac{1}{\gamma-1} \left(\gamma p_0 \overline{u}_{r2} - \frac{1}{r} \int_0^r r' (\overline{T}_1 \overline{S} + \overline{\tilde{T}_1 \tilde{S}}) dr' \right). \tag{36}$$

The only new quantity we need to calculate is $\overline{u}_{r2}$,. This is obtained from Eq. (13)

$$\overline{u}_{r2} = -\frac{\overline{n}_1 \overline{u}_{r1} + \overline{\tilde{n}_1 \tilde{u}_{r1}}}{n_0}. \tag{37}$$

Using Eq. (37) into Eq. (36) together with Eqs. (30), (32), and (6), the second-order change in the average central temperature can be written as

$$\overline{T}_2(0) \equiv \overline{T}_{2_a} + \overline{T}_{2_f}, \tag{38}$$

where

$$\overline{T}_{2_a} = \int_0^a \frac{dr}{\kappa} \left(r \ln\left(\frac{a}{r}\right) \left(\overline{u}_{r1} \frac{dp_1}{dr} + \frac{\overline{T}_1 \overline{S}}{\gamma-1} \right) - 2\gamma n_0 \overline{T}_1 \overline{u}_{r1} \right), \tag{39}$$

$$\overline{T}_{2_f} = -\frac{1}{(\gamma-1)} \int_0^a \frac{dr}{\kappa} \overline{\tilde{T}_1 \tilde{S}} \left(\frac{2\gamma n_0}{D(r)} - r \ln\left(\frac{a}{r}\right) \right). \tag{40}$$

Here, $\overline{T}_{2_a}$ is the change in average temperature from the average radial flows while $\overline{T}_{2_f}$ is the change from the convective cells. $D(r)$ is defined in Eq. (33). To get the expressions in the above form, we switched the order of two integrals to transform the double integral into a single one.

We observe from Eq. (40) that the central temperature is indeed affected by the fluctuating variables. In particular, since $\tilde{u}_{r1}$ depends on $\tilde{S}$, these convection cells influence the central temperature. As discussed earlier, it is to be expected that even closed convection cells will transport heat since a certain amount of heat always leaks out of the flux tubes in a closed loop. What is somewhat unexpected from the result in Eq. (40), however, is that the direction of the convective heat flux could be either up or down the ambient temperature gradient, dT_0/dr, i.e., convective cells could lower or raise the central temperature. To illustrate this aspect, consider the simple limit where the particles are sourced in cold. In that case, the second term in the integrand of Eq. (40) can be ignored and the last term on the right hand side of Eq. (29) can be ignored. Let us also assume $\widetilde{H}=0$, in which case the only term on the right-hand side of Eq. (29) is the first term. If we now examine Eq. (29), we note that the operator ∇^2 in that equation introduces a phase difference of π between $\widetilde{T}_1$ and $\widetilde{S}$ on the right-hand side (at least for sinusoidal type $\widetilde{S}$). This phase difference, together with the fact that the factor $D(r)$ is always positive, due to the stability criterion, indicates that $\widetilde{T}_1$ is proportional to $-\widetilde{S}$. We now insert this information into the first term in the integrand in Eq. (40) and see that $\overline{T_{2_f}}$ is proportional to $+\overline{\widetilde{S}\widetilde{S}}$. Thus, the central temperature goes up in response to convection cells.

In general, however, whether the convective heat flux is positive or negative, with attendant decreases or increases in $\overline{T_2}$, depends on the detailed distribution of the sources. From simple calculations done for $H=0$, we have shown that particle sources peaking strongly near $r=0$ tend to decrease the central temperature whereas sources peaking near $r=a$ tend to raise the central temperature. While these are simple calculations, they nevertheless introduce the possibility that convective cells do not necessarily always transport heat outwards—in principle, appropriate placement of the sources could result in inward heat fluxes. [It is important to add the caveat that these conclusions are arrived at by assuming small S and $\widetilde{H}$: The validity of these inferences in the general case with arbitrary sources should be checked numerically.]

In summary, from this simple case with small sources, we conclude the following: Particle sources generate an average radial flow, $\overline{u_r}$, and "fluctuating" convective flows, $\widetilde{u}_r$ and $\widetilde{u}_z$. The former is driven if there is a nonzero average particle source, i.e., $\overline{S}$. The latter is driven by the axially nonsymmetric piece $\widetilde{S}$. These source-driven flows, first, take the pressure $p(r)$ away from being entirely flat. In addition, they contribute to the heat fluxes. In particular, the fluctuating convective flows may raise or lower the central temperature depending on the details of $\widetilde{S}(r,z)$.

B. General considerations

The conclusions reached from the small particle source calculation above are interesting enough that a general source calculation is in order. This would have to be done numerically, which is not too difficult a task since the system is two-dimensional (2D) and the convection structure is long wavelength and not turbulent if the magnetohydrodynamic (MHD) stability limit is not crossed. As is evident from the small S calculation above, the general case is difficult to solve analytically. We conclude our study here by listing some important features of and omissions in the foregoing calculations:

(a) In the subcritical case, where we are below the MHD stability limit, convection cells tend to take up a structure that is in accordance with the placement of particle sources. This is to say that, in general, plasma flows from regions where it is ionized to regions where it recombines or is extracted from the system. In particular, it can be concluded generally that convection cells are present only in regions where $S\neq 0$: Thus, for localized sources, the cells are also localized.

(b) We have concluded that if $S=0$, there are no convection cells, for arbitrary H. This conclusion needs two caveats: First, is the diamagnetic flows, discussed further in (d) below; second, is that there may be other solutions. To elaborate on the latter, we have found in this paper one solution that satisfies the equations with $\boldsymbol{u}=0$ if $S=0$ (see Sec. III); that, however, does not preclude a solution with $\boldsymbol{u}$ nonzero, since the system is nonlinear. This possibility could be checked numerically.

(c) We reiterate that our quasistatic equations are valid provided the system is stable to the ideal interchange (see discussion in Sec. II). The stability criterion for the interchange is well-known[8] to be

$$\frac{p'}{p}+\frac{\gamma B}{r}\left(\frac{r}{B}\right)'>0. \tag{41}$$

If this is violated, then clearly the quasistatic assumption is inapplicable. Interestingly, this factor appears as a factor multiplying the $\widetilde{u}_r$ term in Eq. (16). It can be checked that, when this factor goes to zero, $\widetilde{u}_r$ becomes singular. Thus, as marginal stability is approached, the convective flows become larger. This is an interesting result that indicates a sub-critical convection phenomenon. More work is needed to check the implications of this for experiment. For example, deviations from quasistatic conditions may cap the increase in the flow from nonlinear effects. From a theoretical viewpoint, it is interesting that a 2D, quasistatic, sub-critical calculation should yield information on the critical stability limit. Standard methods of finding stability boundaries include a full, linear normal mode calculation, or an energy principle approach.[8] Our present method of finding stability limits constitutes another way and may find useful application in stability studies.

(d) One shortcoming of this paper is we have ignored ion diamagnetic flows. The diamagnetic flow is smaller than the sound speed by a factor of ρ_i/a, where ρ_i is the ion gyroradius and a is the crossfield scale size. The convective flows calculated in this paper are of order aS/n, where S is the particle source. Thus, strictly speaking, our conclusions are valid for diamagnetic flows smaller than the source rates. In the course of this work, we have made a preliminary attempt to include diamagnetic effects. This means including the pressure terms in Eq. (5) as well as the inertial and gyroviscous terms in the momentum equation, Eq. (2), taken to first order. This attempt revealed two things: First, $\overline{u_z}$, the average flow in the z-direction, has to be reintroduced as a variable to be determined. We found then that the diamagnetic effects are important largely to determine this average flow but do not influence the source-driven convective flow conclusions already found. Second, and more important, we noted that the derivation of the Braginskii gyroviscosity[10] involves some ordering assumptions that have to be taken into account before the stress tensor is applied to our problem. Specifically, the Braginskii equations are derived under the assumption that the $\boldsymbol{E}\times\boldsymbol{B}$ flows are of order the thermal speed and dominant over all other flows, including diamagnetic flows (by definition) and source driven resistive flows. Thus, strictly, one should only insert $\boldsymbol{E}\times\boldsymbol{B}$ flows into Braginskii's gyroviscous stress. When we used Braginskii's stress tensor and introduced the full flow (in particular the source driven flow) instead of just the $\boldsymbol{E}\times\boldsymbol{B}$ flow, we found results that did not lend themselves to ready physical interpretation, casting doubt on the Braginskii expressions for the stress in this ordering. We checked the kinetic theory and concluded that Braginskii's stress tensor was not the correct one to use for diamagnetic level flows. In fact, Hazeltine and Meiss[11] have derived a form for the viscous stress that is valid for electric and diamagnetic flows of the same order. This is likely the correct stress tensor to be used for our present problem. We have thus concluded that using Braginskii's viscous stress is not a reliable way to understand the role of diamagnetic effects and have decided to relegate this aspect to a separate report.

As mentioned in the Introduction, Kesner and Garnier[1] have recently studied the convective cell problem. Their study is undertaken in dipole geometry and they have included diamagnetic flows, with the Hazeltine–Meiss gyroviscosity. They have solved for an equilibrium that includes convection cells. The equilibrium includes a pressure variation in the azimuthal direction. A particle source is included but does not feature in the final results of the study; a heat source is not explicitly included. Thus our finding that cells are driven only by particle sources cannot be corroborated; from our preliminary work, it does not appear that the diamagnetic flow or dipole geometry would mitigate this finding. Another finding of our study is that the plasma pressure is constant in the axial direction. This comes from lowest order force balance as in Eqs. (7) and (8). Our approach is to use the quasistatic force balance equation, $\nabla p=\boldsymbol{J}\times\boldsymbol{B}$, as the lowest order equation that determines pressure and B variations, and then calculate details of the n and T profiles from the continuity and heat equations. In the Kesner–Garnier study, there is a pressure variation in the azimuthal direction (corresponding to our axial direction). This pressure variation is determined by components of the force balance equation wherein the inertial and gyroviscous terms are included in the same order as the ∇p or the $\boldsymbol{J}\times\boldsymbol{B}$ terms. Thus, the ordering assumptions of the two studies somewhat differ and a detailed comparison of the pressure variation is not possible.

V. CONCLUSION

We have considered a Z-pinch plasma with particle and heat sources. For simplicity, we envision a current-carrying rod down the axis that creates the azimuthal magnetic field. The plasma is in some annulus and the particle and heat sources, S and H, are arbitrarily placed. The sourcing is done quasistatically so that the plasma is always in force balance. In addition, since the parallel sound and heat conduction rates are very fast, the system maintains azimuthal symmetry —thus we assume that S and H are azimuthally symmetric.

We assume that the system is ideal MHD stable (else the quasistatic assumption fails). The operative MHD instability in the system is the interchange instability. If the pressure profiles are gentle, the system is known to be stable to the interchange. The stability criterion is given by Eq. (41). Such a stable system, when sourced asymmetrically, will, in general, develop low-level convection. Our objective herein is to study this convection.

We have formulated the quasistatic equations to describe this problem in this paper. The equations are sufficiently complex, however, that it is difficult to arrive at conclusions in the general case using analytic methods. We have consequently made two simplifying assumptions: We have neglected ion diamagnetic flows, and we have (in part) assumed a weak particle source. The first assumption was made because we have found that Braginskii's gyroviscous stress is derived using ordering assumptions that do not apply to our problem. We plan to correct this ordering limitation before using the gyroviscous stress in a followup work. The second assumption has made the system more tractable. It has helped us draw some conclusions that, while based on the weak source ordering, may nonetheless be of greater generality.

Within these qualifications, our findings in this paper are as follows:

(1) Under all conditions, the pressure stays independent of z, i.e., $p=p(r)$. This is a consequence of the well-known MHD theorem that says that for a purely azimuthal magnetic field, an equilibrium cannot exist wherein the pressure varies axially. This is a powerful constraint: in contrast, for example, in a plasma in a uniform magnetic field, asymmetric sourcing would not result in uniform pressure in the direction of asymmetry.

(2) If there are no particle sources, there is no convection: A solution exists with all flow being zero. This is the case for an arbitrary heat source. Thus, the placement of heaters for an experiment should not be a concern as far as convective losses go. By the same token, the distribution of particle sources is a matter of concern.

(3) Particle sources result in convection. In general, there is an average radial flow and superimposed are some fluctuating convective flows with variation in the z-direction.

(4) The heat transport is made up of three pieces: Heat flux from the usual thermal conduction, heat flux from the average radial convection, and, finally, heat flux from the convective flows. If we assume that particles are sourced in at low temperature somewhere inside the plasma and extracted at the edge of the plasma (such that the average

source rate is zero), then the average radial convection is outward and tends to reduce the central temperature from a combination of outward heat convection as well as adiabatic cooling. The heat flux from the third component, the convective flows, is more complicated. In some cases, this heat flux can even be negative: that is to say, the central temperature may drop or rise as a result of this convective pattern. The precise effect on the central temperature depends on the details of the sourcing function.

ACKNOWLEDGMENTS

We acknowledge useful discussions with Dr. J. Kesner and Dr. M. Mauel.

This work was supported in part by a grant from the U.S. Department of Energy. A.M.R. was also supported in part by a fellowship from the University of Maryland Department of Physics.

[1] J. Kesner and D. T. Garnier, Phys. Plasmas **7**, 2733 (2000).
[2] J. R. Drake, J. R. Greenwood, G. A. Navratil, and R. S. Post, Phys. Fluids **20**, 148 (1977).
[3] G. A. Navratil, R. S. Post, and A. Butcher Ehrhardt, Phys. Fluids **20**, 156 (1977).
[4] A. Butcher Ehrhardt, H. R. Garner, G. A. Navratil, and R. S. Post, Phys. Fluids **24**, 1859 (1981).
[5] J. R. Drake and J. G. Berryman, Phys. Fluids **20**, 851 (1977).
[6] H. Okuda and J. M. Dawson, Phys. Fluids **16**, 408 (1973).
[7] A. B. Hassam and R. M. Kulsrud, Phys. Fluids **22**, 2097 (1979).
[8] R. J. Goldston and P. H. Rutherford, *Introduction to Plasma Physics* (Institute of Physics, Philadelphia, 1995).
[9] D. Garnier, J. Kesner, and M. E. Mauel, Phys. Plasmas **6**, 3431 (1999).
[10] S. I. Braginskii, *Reviews of Plasma Physics*, edited by M.A. Leontovich (Consultants Bureau, New York, 1965), Vol. 1, p. 205.
[11] R. D. Hazeltine and J. D. Meiss, *Plasma Confinement* (Addison-Wesley, Redwood City, 1992).